Modeling User Responses to Pricing: Simultaneous Route and Departure Time Network Equilibrium with Heterogeneous Users

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Chung-Cheng Lu and Hani S. Mahmassani

ABSTRACT

An important dimension in users' responses to dynamic pricing schemes is the possible shift in the timing of trips, in addition to changing travel routes. User responses to pricing are governed by individual tripmakers' preferences, such as their value of time, and the cost they attach to late vs. early arrival relative to the destination. These behavioral characteristics vary across users. Capturing the heterogeneity of users in this regard is important in predicting the impact of dynamic pricing schemes. While previous models have been proposed to find simultaneous route and departure time user equilibrium (SRDUE) in a network, these have not considered the heterogeneity of users in the underlying departure time and path choice decision framework. In a previous contribution, the authors have shown how to incorporate user heterogeneity in determining equilibrium route choices in a network in response to pricing. This paper presents a generalization of that framework to incorporate joint consideration of route and departure time, as well as heterogeneity in a wider range of behavioral characteristics. A model for a multi-criterion SRDUE (or MSRDUE) is presented, along with a simulation-based solution algorithm intended for practical network application. The model explicitly considers heterogeneous users with different values of time (VOT) and values of (early or late) schedule delay (VOESD or VOLSD), in their joint choice of departure times and paths characterized by a set of trip attributes that include travel time, out-of-pocket cost, and schedule delay cost. The MSRDUE problem is formulated as an infinite dimensional variational inequality (VI) problem and solved by a column generation-based algorithmic framework that embeds (i) an extreme non-dominated alternative-finding algorithm to obtain the VOT, VOESD, and VOLSD breakpoints that define multiple user classes, and the associated least trip cost (joint departure time and path) alternative for each user class, (ii) a traffic simulator to capture traffic flow dynamics and determine experienced travel costs; and (iii) a path swapping multi-class alternative flow updating scheme to solve the restricted multi-class SRDUE problem defined by a subset of feasible alternatives. Application to an actual network illustrates the properties of the algorithm, and underscores the importance of capturing user heterogeneity and temporal shifts in the appraisal of dynamic pricing schemes.

Keywords: user responses to pricing; dynamic traffic assignment; simultaneous route and departure time user equilibrium; dynamic road pricing; user heterogeneity; multi-criterion time-dependent dynamic network equilibrium.

1 INTRODUCTION

To support the planning, operation, and evaluation of dynamic road pricing strategies, a simulation-based bi-criterion dynamic (or time-dependent) user equilibrium (BDUE) model, which explicitly recognizes user heterogeneity in value of time (VOT) in the underlying path choice process, was recently proposed by Lu et al. (*1*). While the BDUE model is able to realistically capture users' path choices in response to time-varying toll charges, it assumes that the time-varying OD demands for the entire feasible range of VOT and over the planning horizon are known, a priori; or equivalently that trip-makers' departure times are fixed. However, in general, a trip-maker facing a toll road with time-varying charges would not only change path but also adjust departure time so as to minimize his/her total trip cost. Some theoretical studies using analytical approaches, e.g., Arnott et al. (*2*), have further found that time-varying tolls generally yield greater efficiency gains than static tolls because the former reduce queueing delays by altering travelers' departure times rather than simply their paths. Therefore, a realistic extension of the BDUE model for application to the evaluation of road pricing schemes is to capture trip-makers' departure time choices, in addition to path choices, in response to time-varying toll charges.

This paper presents a simulation-based multi-criterion simultaneous route and departure time user equilibrium (MSRDUE) model, and associated solution algorithm, which explicitly considers heterogeneous users with different values of time (VOT) and values of (early or late) schedule delay (VOESD or VOLSD), simultaneously choosing departure times and paths characterized by the following trip attributes: travel time, out-of-pocket cost, and schedule delay cost, where schedule delay is defined as the difference between actual and preferred arrival times (PAT). Several SRDUE models, analytical and simulation-based, have been proposed in the literature (e.g. *3*, *4*, *5*, *6*, *7*, 8, 9), but none have considered heterogeneous users with different VOT, VOESD, and VOLSD. The latter adds considerable algorithmic and computational complexity to the problem, yet is essential for realistic representation of user responses to the kinds of pricing schemes of interest to policy makers and public/private toll operators.

Following the behavioral modeling framework typically adopted in the above (deterministic) SRDUE models for describing trip-makers' joint departure time and path choice behavior, each trip-maker is assumed to choose the combined departure time (interval) and path *alternative*, which minimizes his/her *trip cost*, defined as the sum of travel cost, travel time weighted by VOT, and early or late schedule delay weighted by VOESD or VOLSD. The MSDUE problem is formulated as an infinite-dimensional variational inequality (VI) problem, and solved by a column generation-based algorithmic framework that embeds (i) an extreme non-dominated alternative finding algorithm – SPAM (Sequential Parametric Analysis Method) to obtain the VOT, VOESD, and VOLSD breakpoints that define multiple user classes, and determine the least trip cost alternative for each user class, (ii) a traffic simulator - DYNASMART (*10*) to capture traffic flow dynamics and determine experienced travel costs; and (iii) a path swapping-based multi-class alternative flow updating (or equilibrating) scheme to solve the restricted multi-class SRDUE (RMCSRDUE) problem defined by a subset of feasible alternatives.

This paper is structured as follows. Section 2 presents the problem statement and related assumptions and definitions, followed by the VI formulation of the MSRDUE problem in section 3. In section 4, the column generation-based solution algorithm is described; it consists mainly of the SPAM and the multi-class alternative flow equilibration scheme. Section 5 reports the experimental results illustrating the convergence of the algorithm and how user heterogeneity affects the departure time and path flow patterns as well as toll road usage under dynamic road pricing scenarios. Section 6 concludes the paper.

2 ASSUMPTIONS, DEFINITIONS, AND PROBLEM STATEMENT

Consider a network G = (N, A), where N is the set of nodes and A is the set of directed links (i, j), $i \in N$ and $j \in N$. The time period of interest (planning horizon) is discretized into a set of small time intervals, $S = \{t_0,\ t_0 + \sigma,\ t_0 + 2\sigma, \ldots,\ t_0 + M\sigma\}$, where t_0 is the earliest possible departure time from any origin node, σ a small time interval during which no perceptible changes in traffic conditions and/or travel cost occur, and M a large number such that the intervals from t_0 to $t_0 + M\sigma$ cover S. Without loss of generality, associated with each arc (i, j) and time interval t are two essential time-dependent arc travel impedances: time ($d_{ij}(t)$) and cost ($c_{ij}(t)$), which are incurred in travel from node i, in time interval t, to node j. The link generalized travel cost perceived by a trip-maker with VOT α from node i in time interval t to node j is defined as:

$$g_{ij}(t) = c_{ij}(t) + \alpha \times d_{ij}(t) \tag{1}$$

The VOT represents how much money a trip-maker is willing to trade for a unit time saving. Presented below are the other notations and variables used in this chapter.

o	index for an origin node, $o = 1,\ldots, O$.
d	index for a destination node, $d = 1,\ldots, D$.
τ	index for a departure time interval, $\tau = 1,...,T1$.
θ	index for a preferred arrival time (PAT) interval, $\theta = 1,...,T2$.
α	value of time (VOT), $\alpha \in [\alpha^{\min}, \alpha^{\max}]$.
β	value of early schedule delay (VOESD), $\beta \in [\beta^{\min}, \beta^{\max}]$.
λ	value of late schedule delay (VOLSD), $\lambda \in [\lambda^{\min}, \lambda^{\max}]$.
$P(o,d)$	the set of feasible paths for a given origin-destination (OD) pair, (o,d).
p	index for a path $p \in P(o,d)$.
$h_{od}(\theta,\alpha,\beta,\lambda)$	number of trips (users) with a possible combination of (α, β, λ), traveling from o to d, and expecting to arrive in time interval θ; (part of given demand input).
$r^{\tau}_{odp}(\theta,\alpha,\beta,\lambda)$	number of trips (users) with a possible combination of (α, β, λ), traveling from o to d by alternative (τ, p), leaving in time interval τ along path p, and expecting to arrive in PAT interval θ; these are the unknown decision variables.
$r(\theta,\alpha,\beta,\lambda)$	class-specific alternative flow vector for a given combination of θ, α; β, and λ; $r(\theta,\alpha,\beta,\lambda) = \{r^{\tau}_{odp}(\theta,\alpha,\beta,\lambda), \forall (o,d),(\tau,p)\}$.
r	multi-class alternative flow vector for all OD pairs and all possible values of θ, α; β, and λ; $r = \{r(\theta,\alpha,\beta,\lambda), \forall \theta,\alpha,\beta,\lambda\}$.
TT^{τ}_{odp}	average (or unit) experienced travel time for the trips traveling from o to d by alternative (τ, p).
TT	vector of experienced travel times; ***TT*** $= \{TT^{\tau}_{odp}, \forall o, d, \tau$, and $p \in P(o,d)\}$.
TC^{τ}_{odp}	average (or unit) experienced travel cost (i.e. road toll) for the trips traveling from o to d by alternative (τ, p).
TC	vector of experienced travel costs; ***TC*** $= \{TC^{\tau}_{odp}, \forall o, d, \tau$, and $p \in P(o,d)\}$.
$\varphi^{\tau}_{odp}(\theta)$	experienced schedule delay (or arrival) cost for the trips traveling from o to d by alternative (τ, p) with PAT interval θ.

The schedule delay cost $\varphi^{\tau}_{odp}(\theta)$ is determined according to the piece-wise linear function:

$$\varphi^{\tau}_{odp}(\theta) = \begin{cases} \beta \times [\theta^{lb} - (\tau + TT^{\tau}_{odp})], \text{if } \theta^{lb} > \tau + TT^{\tau}_{odp}, \\ 0, \text{if } \theta^{lb} \le \tau + TT^{\tau}_{odp} \le \theta^{ub}, \\ \lambda \times [(\tau + TT^{\tau}_{odp}) - \theta^{ub}], \text{if } \theta^{ub} < \tau + TT^{\tau}_{odp}, \end{cases} \tag{2}$$

where $[\theta^{lb}, \theta^{ub}]$ is the range of a PAT interval θ and $\tau + TT^{\tau}_{odp}$ is the arrival time from o to d via the alternative (τ, p). This schedule delay cost function assumes that travelers incur no arrival time cost (penalty) if their arrival times are in the acceptable range $[\theta^{lb}, \theta^{ub}]$. The (average or unit) trip cost perceived and experienced by the travelers with the same (θ, α, β, λ) traveling from origin o to destination d using alternative (τ, p) is defined as the sum of the perceived path generalized cost ($TC^{\tau}_{odp} + \alpha \times TT^{\tau}_{odp}$) and the schedule delay cost (Eq.(2)) associated with that alternative:

$$G^{\tau}_{odp}(r;\theta,\alpha,\beta,\lambda) = TC^{\tau}_{odp} + \alpha \times TT^{\tau}_{odp} + \varphi^{\tau}_{odp}(\theta), \qquad (3)$$

where $TT^{\tau}_{odp} = \sum_{(i,j,t)\in p} d_{ij}(t)$ and $TC^{\tau}_{odp} = \sum_{(i,j,t)\in p} c_{ij}(t)$. Note that this trip cost is evaluated at a multi-class alternative flow vector r to take into account the interactions of heterogeneous users in a network. It is clear that Eq.(3) can be expanded as follows, by incorporating Eq.(2):

$$G^{\tau}_{odp}(r;\theta,\alpha,\beta,\lambda) = TC^{\tau}_{odp} + \alpha \times TT^{\tau}_{odp} + \beta \times ESD^{\tau}_{odp}(\theta) + \lambda \times LSD^{\tau}_{odp}(\theta) \qquad (4)$$

where $ESD^{\tau}_{odp}(\theta) = \max\{0, \theta^{lb} - (\tau + TT^{\tau}_{odp})\}$ and $LSD^{\tau}_{odp}(\theta) = \max\{0, (\tau + TT^{\tau}_{odp}) - \theta^{ub}\}$ are the early and late schedule delays, respectively, with respect to the PAT interval θ.

To explicitly consider heterogeneity of the population, VOT (α), VOESD (β), and VOLASD (λ) in this study are assumed to be continuous random variables distributed across the population of trips, with the probability density functions:

$\phi(\alpha)>0$, $\forall \alpha \in [\alpha^{\min}, \alpha^{\max}]$ and $\int_{\alpha^{\min}}^{\alpha^{\max}} \phi(\alpha) d\alpha = 1$,

$\phi(\beta)>0$, $\forall \beta \in [\beta^{\min}, \beta^{\max}]$ and $\int_{\beta^{\min}}^{\beta^{\max}} \phi(\beta) d\beta = 1$, and

$\phi(\lambda)>0$, $\forall \lambda \in [\lambda^{\min}, \lambda^{\max}]$ and $\int_{\lambda^{\min}}^{\lambda^{\max}} \phi(\lambda) d\lambda = 1$.

Note that the distributions of VOT, VOESD, and VOLSD, which could be estimated from survey data (e.g., *11*) or loop detector data (e.g., *12*), are assumed to be known and given a priori. In general, following the empirical results reported by Small (*13*), it is assumed that $\lambda > \alpha > \beta > 0$, for all trip-makers in a network; that is, trip-makers value the cost of LSD higher than the costs of time and ESD. Note that since these parameters are continuously distributed, theoretically, to obtain the number of trips with a specific combination of α, β and λ, integrals should be applied on the infinitesimal intervals of these variables with their density functions. For ease of presentation, we denote $h_{od}(\theta,\alpha,\beta,\lambda) \equiv \int_{\alpha}^{\alpha+\Delta\alpha}\int_{\beta}^{\beta+\Delta\beta}\int_{\lambda}^{\lambda+\Delta\lambda} h_{od}(\theta,\alpha,\beta,\lambda)d\lambda d\beta d\alpha$. This simplified notation is also applied in the definition of $r_{odp}^{\tau}(\theta,\alpha,\beta,\lambda)$ and $r(\theta,\alpha,\beta,\lambda)$.

Additionally, this study allows each trip to have its own PAT interval θ by assuming the PAT pattern follows a given discrete distribution with the probability mass function:

$$\varpi(\theta)>0,\ \forall\theta = 1, \ldots, T, \text{ and } \sum_{\theta=1}^{T2}\varpi(\theta) = 1$$

The travel demands for each OD pair (o, d), every PAT θ and the entire ranges of VOT, VOESD, and VOLSD over the planning horizon (i.e. $h_{od}(\theta,\alpha,\beta,\lambda)$) are assumed to be known and given a priori.

The behavioral assumption made in this study is that each trip-maker would choose the alternative (i.e. combination of arrival time and path) that minimizes his or her trip cost, defined in Eq.(4). Specifically, for trips with the same (o, d, θ, α, β, λ), an alternative (τ^*, p^*) will be selected if and only if

$$(\tau^*, p^*) = \arg\min_{\forall(\tau,p)} G_{odp}^{\tau}(r;\theta,\alpha,\beta,\lambda).$$

As stated, this assumption connotes that a deterministic path choice approach is applied, though an extension to the probabilistic (or stochastic) case could readily be accommodated by adding a random error term to the cost function (Eq. 3) or any of its components, reflecting travelers' perception errors on travel times and/or directly on the overall cost (disutility). This extension would result in a logit- or probit-based choice model.

Based on the above assumption, the multi-criterion simultaneous route and departure time user equilibrium (MSRDUE), a multi-criterion and dynamic extension of Wardrop's first principle (*14*), is defined as follows.

Definition 1: MSRDUE

For each OD pair, no trip (user) can decrease its experienced trip cost, with respect to that trip's particular VOT, VOESD, VOLSD, and PAT interval, by unilaterally changing departure time and/or path.

This implies that, at MSRDUE, each trip-maker is assigned to the alternative that has the least trip cost with respect to his/her own PAT, VOT, VOESD, and VOLSD. This definition can also be viewed as the heterogeneous (or multi-criterion) generalization of the simultaneous route and departure time user equilibrium (SRDUE) in the literature (e.g., *3, 4, 7*). Based on the above definition, the MSRDUE conditions can be mathematically stated as: $\forall \theta, \alpha, \beta, \lambda$, and $\forall o, d$

$$r_{odp}^{\tau} * (\theta, \alpha, \beta, \lambda)[G_{odp}^{\tau}(r^*; \theta, \alpha, \beta, \lambda) - \pi_{od}(r^*; \theta, \alpha, \beta, \lambda)] = 0, \ \forall (\tau, p), \tag{5}$$

$$G_{odp}^{\tau}(r^*; \theta, \alpha, \beta, \lambda) - \pi_{od}(r^*; \theta, \alpha, \beta, \lambda) \geq 0, \ \forall (\tau, p), \tag{6}$$

$$\sum_{\tau=1}^{T} \sum_{p \in P(o,d)} r_{odp}^{\tau}(\theta, \alpha, \beta, \lambda) = h_{od}(\theta, \alpha, \beta, \lambda), \tag{7}$$

$$r_{odp}^{\tau}(\theta, \alpha, \beta, \lambda) \geq 0, \ \forall (\tau, p), \tag{8}$$

where $r^* = \{r^*(\theta, \alpha, \beta, \lambda), \forall \theta, \alpha, \beta, \lambda\}$ is a multi-class MSRDUE alternative flow vector, and $\pi_{od}(r^*; \theta, \alpha, \beta, \lambda)$ is the minimum OD trip cost, evaluated at r^*, for the trips with the same $(o, d, \theta, \alpha, \beta, \lambda)$. Given the assumptions and definition above, this study aims at solving the MSRDUE problem, under a given set of time-varying link tolls and given heterogeneous OD demands, to obtain temporal splits (among departure times) and spatial distributions (over paths) satisfying the MSRDUE conditions. Specifically, the focus is on determining the MSRDUE alternative flows: $r_{odp}^{\tau}(\theta, \alpha, \beta, \lambda)$, $\forall o, d, \theta, \alpha, \beta, \lambda$, and (τ, p) in a vehicular network.

3 AN INFINITE-DIMENSIONAL VI FORMULATION OF THE MSRDUE

Let $\Omega(\theta, \alpha, \beta, \lambda) \equiv \{ r(\theta, \alpha, \beta, \lambda) \}$ be the set of feasible class-specific alternative flow vectors satisfying the OD flow conservation constraints (7) and non-negativity constraints (8). Solving for the MSRDUE (joint departure time and route) alternative flow pattern r^* is equivalent to finding the solution of a system of variational inequalities: $\forall \theta, \alpha, \beta$, and λ,

find $r^*(\theta, \alpha, \beta, \lambda) \in \Omega(\theta, \alpha, \beta, \lambda)$, such that

$$\sum_{o=1}^{O} \sum_{d=1}^{D} \sum_{\tau=1}^{T1} \sum_{p \in P(o,d)} G_{odp}^{\tau}(r^*; \theta, \alpha, \beta, \lambda)[r_{odp}^{\tau}(\theta, \alpha, \beta, \lambda) - r_{odp}^{\tau} * (\theta, \alpha, \beta, \lambda)] \geq 0 \tag{9}$$

or in the following vector form for simplicity and clarity,

$$G(r^*; \theta, \alpha, \beta, \lambda) \circ [r(\theta, \alpha, \beta, \lambda) - r^*(\theta, \alpha, \beta, \lambda)] \geq 0,$$
$$\forall r(\theta, \alpha, \beta, \lambda) \in \Omega(\theta, \alpha, \beta, \lambda), \tag{10}$$

where $\circ$ denotes the inner product of two vectors with the same size. Eq. (10) can be further restated as the following infinite-dimensional VI (e.g., *15*):

Find $r^* \equiv \{ r^*(\theta,\alpha,\beta,\lambda)\,, \forall\, \theta,\alpha,\beta, \text{and } \lambda \}$ and $r^* \in \Omega$ such that

$$G(r^*)^T \circ (r - r^*) \geq 0,\ \forall\ r \in \Omega \qquad (11)$$

where $G(r^*) \equiv \{ G(r^*;\theta,\alpha,\beta,\lambda)\,,\ \forall\, \theta,\alpha,\beta, \text{and } \lambda \}$ and $\Omega = \{r\}$, the set of feasible multi-class alternative flow vectors. Note that the vectors: $G(r^*)$ and r^* (or r) have the same (possibly infinite) number of elements.

4 MSRDUE ALGORITHM

4.1 Overview of the column generation-based algorithmic framework

Since the MSRDUE problem of interest seeks a network equilibrium in terms of (joint departure time and route alternative) trip costs of network users, a set of feasible alternatives on which the given heterogeneous OD demands are to be equilibrated is required for MSRDUE solution algorithms. It is generally very difficult, if not impossible, to enumerate the complete set of feasible alternatives for all OD pairs and all possible PAT, VOT, VOESD, and VOLSD in a road network of practical size. Furthermore, only a (small) fraction of alternatives would carry positive flows in MSRDUE solutions. To avoid explicit enumeration of all possible alternatives, this study applies a column generation-based approach that generates and augments a representative subset of alternatives with competitive trip cost.

The column generation-based approach augments, in the outer loop, the subset of feasible alternatives and solves, in the inner loop, the “restricted” multi-class simultaneous route and departure time user equilibrium (RMCSRDUE) problem, defined by the (current) subset of feasible alternatives. In each outer loop iteration k, the SPAM is applied to (i) obtain the breakpoints which partition the entire ranges of VOT, VOESD, and VOLSD into many subintervals and accordingly determine the corresponding multiple user classes, and (ii) find least trip cost (i.e., extreme non-dominated) alternative for each user class. New alternatives, if any, are added to the current alternative set. The algorithm terminates if no new alternative is found for any user class, or a preset convergence criterion is satisfied; otherwise, the RMCSRDUE problem is solved to equilibrate the heterogeneous OD demands on the current alternative set before returning to the alternative generation step (i.e. outer loop). Solving the RMCSRDUE problem forms the inner loop (with iteration counter l) of the column generation-based framework. It features a multi-class alternative flow updating (or equilibrating) scheme that proceeds iteratively to equilibrium, in a manner similar to the restricted path set equilibration scheme suggested by Larsson and Patriksson (*16*).

This idea of obtaining VOT, VOESD, and VOLSD breakpoints that naturally determine multiple user classes, and solving the RMCSRDUE problem by equilibrating alternative flows for each user class, is based on the following assumption. In the disutility minimization-based departure time and path choice modeling framework with convex disutility (i.e. trip cost) functions, all trips would choose only among the set of extreme efficient (or non-dominated) alternatives, and tripmakers in each user class behave similarly in their departure time and path choices (e.g. *15*, *17*). Also embedded in this algorithmic framework is the traffic simulator – DYNASMART (*10*), which performs multi-class dynamic network loading (MDNL) to determine link travel times and experienced (path) travel times and travel costs for a given multi-class alternative flow pattern r; traffic flow propagation and the vehicular spatial and temporal interactions are addressed through the traffic simulation.

By and large, the original MSRDUE problem is solved in this algorithmic framework as a series of approximate RMCSRDUE problems to progressively find MSRDUE solutions. The column generation-based MSRDUE solution algorithm is described in the following, and its flow chart is presented in Fig. 1.

MSRDUE Solution Algorithm

Initialization

0. Input: (1) heterogeneous demands for each OD pair (o, d), every PAT θ and the entire ranges of VOT, VOESD, and VOLSD over the planning horizon (i.e. $r_{od}(\theta, \alpha, \beta, \lambda)$), (2) time-dependent link tolls, (3) VOT, VOESD, and VOLSD distributions, and (4) initial alternatives and assignment.
1. Set the outer loop iteration counter k = 0. Perform network loading with the traffic simulator to evaluate the initial assignment and obtain time-dependent link travel times and experienced (path) travel times and costs (i.e. $\boldsymbol{TT}$ and $\boldsymbol{TC}$).

Outer Loop – augmenting the alternative set

2. Use the SPAM to obtain the set of (extreme efficient) alternatives, their corresponding least trip costs (π^k) and breakpoints of VOT, VOESD, and VOLSD that define the multiple user classes.

3. Convergence checking: if (a) no new alternative is found or (b) $k = K_{max}$ (maximum number of outer loop iterations) then stop; otherwise start the inner loop (go to step 4).

Inner Loop – solving the RMC-SRDUE problem

4. Set the inner loop iteration counter $l = 0$.
5. Update assignment: determine assignment r^{l+1} by using multi-class alternative flow updating (or equilibrating) scheme. Set $l = l + 1$.
6. Perform a MDNL by the traffic simulator to evaluate the new assignment r^l and obtain experienced path travel times and costs (i.e. ***TT*** and ***TC***).
7. Convergence checking: If the preset convergence threshold is reached or $l = L_{max}$ (maximum number of inner iterations), then set $k = k+1$ and return to step 2 with current link travel times;

4.2 Augmenting the Alternative Set by SPAM

The major hurdle in solving the MSRDUE problem of interest is due to the relaxation of the PAT, VOT, VOESD, and VOLSD from constants to discrete or continuous random variables, and hence the need to find an equilibrium state resulting from the interaction of (possibly infinite) many classes of trips, each of which corresponds to a class-specific combination of (θ, α, β, λ), in a network. If, in the extreme case, each trip-maker (or class) requires its own set of feasible alternatives for all OD pairs, finding and storing such a grand alternative set is computationally intractable and memory intensive in (road) network applications of practical sizes. In order to circumvent the difficulty of finding and storing the least trip cost alternative for each individual trip-maker with different PAT, VOT, VOESD, and VOLSD, the SPAM is proposed to find the set of extreme efficient (or non-dominated) alternative trees, each of which minimizes the parametric trip cost function (Eq. 4) for a particular PAT interval and certain subintervals of VOT, VOESD, and VOLSD. This approach is based on the assumption (e.g. *15*, *17*) that in the disutility minimization-based path choice modeling framework with convex disutility functions, all trips would choose only among the set of extreme efficient paths corresponding to the extreme points on the efficient frontier in the criterion space.

otherwise go back to step 5.

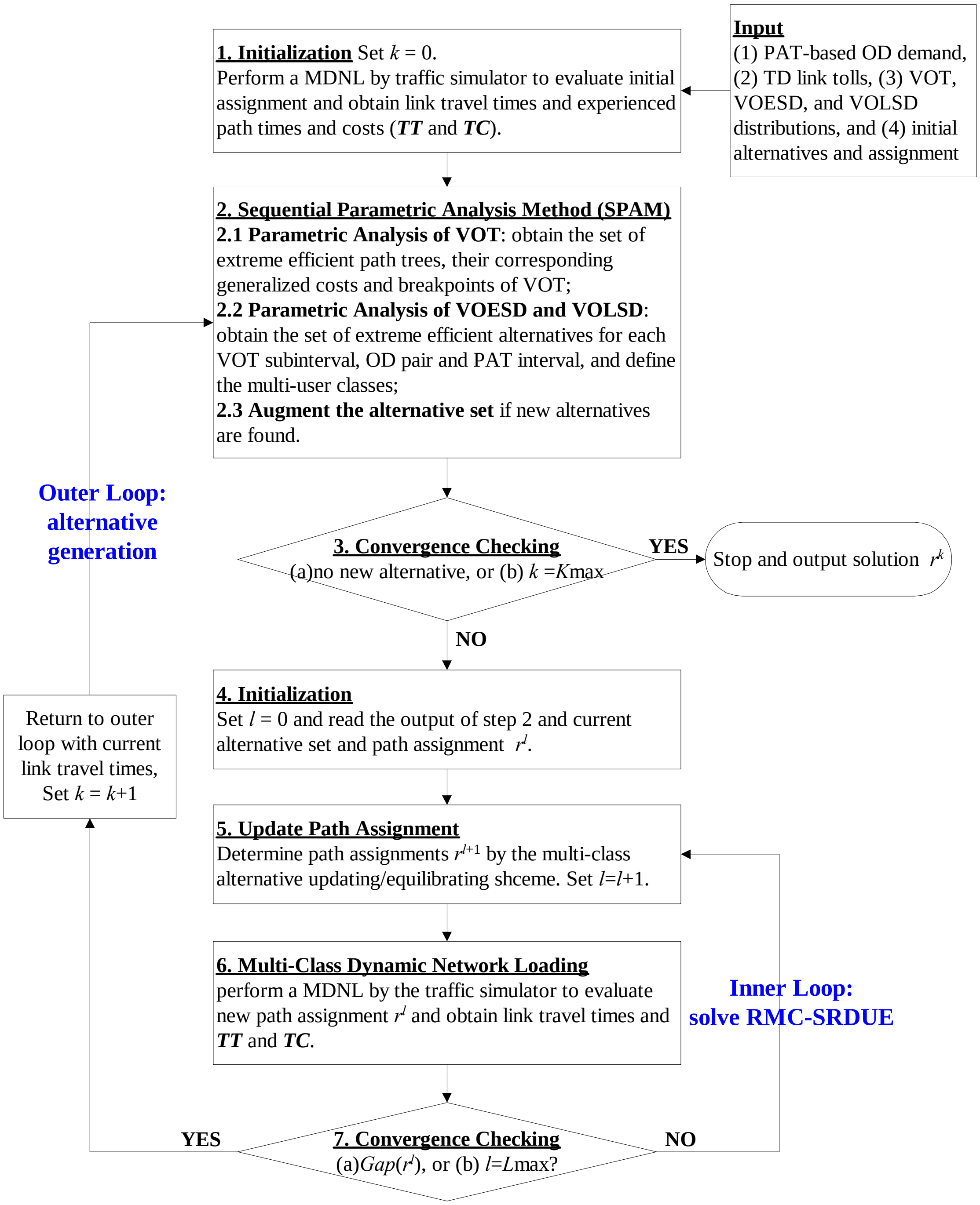


Fig. 1 Flow chart of the MSRDUE solution algorithm

The SPAM consists of two stages: (i) parametric analysis of VOT (α) and (ii) parametric analyses of VOESD (β) and VOLSD (λ) for a given VOT subinterval (Fig. 2). In the first stage, the parametric analysis bi-criterion time-dependent least cost path algorithm, developed by Mahmassani et al. (*18*), is applied to find the set of time-dependent extreme non-dominated path trees and the corresponding set of VOT breakpoints (or subintervals). In the second stage, each of those so-obtained trees (rooted at the same destination but corresponding to different VOT subintervals) is then parametrically analyzed with respect to VOESD and VOLSD to determine VOESD and VOLSD subintervals (and breakpoints) and the least trip cost alternatives for those subintervals. These breakpoints naturally define the multiple user classes used in the subsequent step of the algorithm (i.e. multi-class alternative flow updating). This second stage of the SPAM is conducted on an expanded network by adding a dummy node for each PAT interval and connecting that dummy node with all possible arrival times by artificial links. The theoretical derivations and the detail of the SPAM can be found in Lu (*19*). The above two-stage process is repeated for each destination node. In each iteration k, the SPAM is performed to find the set of extreme efficient alternatives for all OD pairs and the corresponding breakpoints that (i) partition the feasible ranges of VOT, VOESD, and VOLSD and (ii) define the multiple user classes for the RMCSRDUE problem solved in the inner loop of the column generation-based MSRDUE solution algorithm.

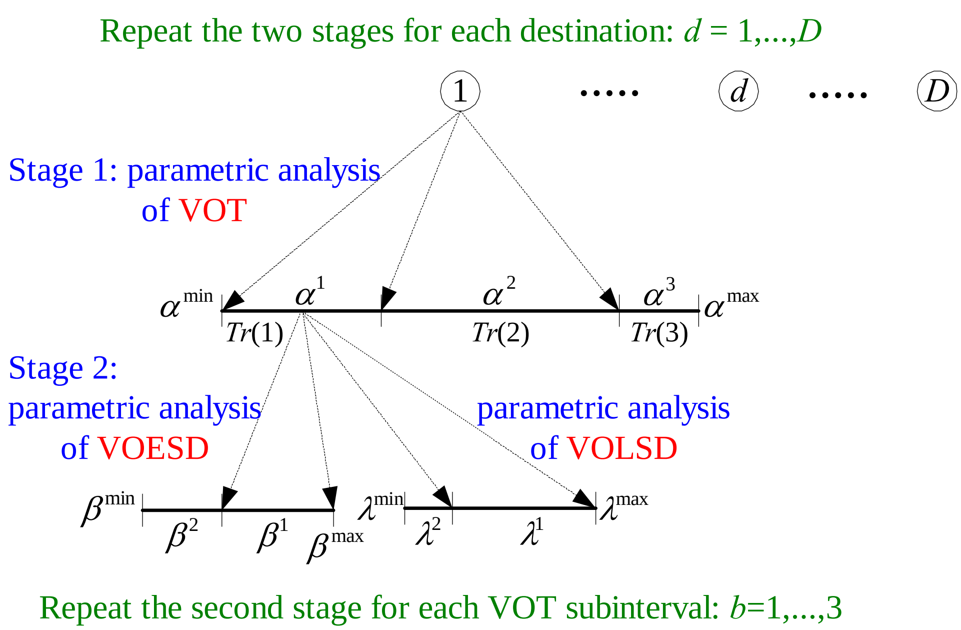


Fig. 2 Sequential Parametric Analysis Method

4.3 Solving the RMCSRDUE Problem

With the breakpoints of VOT, VOESD, and VOLSD determined by the SPAM in an outer loop iteration k of the algorithm, the entire population of heterogeneous trips in a network can be divided into a finite number of user classes, and hence the original (infinite-dimensional) MSRDUE problem of interest can be reduced to the (finite-dimensional) multi-class SRDUE problem, in which equilibration within each user class is sought. Furthermore, since, in each iteration, the multi-class SRDUE is determined based on the current subset of feasible alternatives, the sub-problem solved in the inner loop is named the "restricted" multi-class SRDUE (or RMCSRDUE) problem by following the terminology often adopted in the literature (e.g. *21*). Based on the output of SPAM, for each VOT subinterval b and PAT θ, the corresponding trips are grouped into the user classes:

$$u(b,\theta,m_{\beta(b,\theta)},n_{\lambda(b,\theta)}),\ m=1,...,M(b,\theta),\ n=1,...,N(b,\theta),$$

each of which is defined by a pair VOESD and VOLSD subintervals. $M(b,\theta)$ and $N(b,\theta)$ are the numbers of subintervals (or classes) of VOESD and VOLSD, respectively, corresponding to a combination of (b,θ). Note that this user class notation is simplified as $u(b,\theta,m,n)$ for ease of presentation. Denote by $alt_{od}(b,\theta,m,n)$ the set of alternatives corresponding to the user class $u(b,\theta,m,n)$ and OD pair *od*; we have that $alt_{od}(b,\theta,m,n)=alt_{od}(b,\theta,m_{b,\theta})\cup alt_{od}(b,\theta,n_{b,\theta})$, where $alt_{od}(b,\theta,m_{b,\theta})$ and $alt_{od}(b,\theta,n_{b,\theta})$ are the sets of alternatives corresponding to the VOESD subinterval $[\beta^{m-1},\beta^{m})_{b,\theta}$ and VOLSD subinterval $[\lambda^{n-1},\lambda^{n})_{b,\theta}$, respectively.

Specifically, solving the RMCSRDUE problem consists in finding a (finite-dimensional) multi-class alternative flow vector that satisfies the RMCSRDUE definition: *for each user class* $u(b,\theta,m,n)$ *and each OD pair, no trip can decrease the experienced trip cost by unilaterally changing departure time and/or path*. The following variables and notations are defined (or redefined) for the RMCSRDUE problem.

$r_{odp}^{\tau}(b,\theta,m,n)$	number of trips of user class $u(b,\theta,m,n)$ traveling from *o* to *d* and choosing the alternative $(\tau,p)\in alt_{od}(b,\theta,m,n)$.
r	$\equiv\{r(b,\theta,m,n)\}$; the multi-class path flow vector.
$G_{odp}^{\tau}(r;b,\theta,m,n)$	the trip cost of user class $u(b,\theta,m,n)$ trips traveling from *o* to *d* and choosing the alternative $(\tau,p)\in alt_{od}(b,\theta,m,n)$, evaluated at *r*.
$\pi_{od}(r;b,\theta,m,n)$	the least trip cost of user class $u(b,\theta,m,n)$ trips traveling from *o* to *d* and, evaluated at the assignment *r*.

To solve the RMC-SRDUE problem, this study proposes a multi-class alternative flow updating scheme which decomposes the RMCSRDUE problem into many (b,θ,m,n,o,d) sub-problems and solves each of them by adjusting OD flows between (all) non-least trip cost alternatives and the least trip cost alternative. Let (τ^*,p^*) be the *referenced* least trip cost alternative for the user class $u(b,\theta,m,n)$ and each OD pair (o, d). Specifically, for each (b,θ,m,n,o,d) sub-problem, the multi-class alternative flow updating scheme in an inner loop iteration l is as follows:

$$r_{odp}^{\tau,l+1}(b,\theta,m,n) =$$
$$\max\{0, r_{odp}^{\tau,l}(b,\theta,m,n) - \rho^l \times \frac{r_{odp}^{\tau,l}(b,\theta,m,n) \times \Delta_{odp}^{\tau,l}(b,\theta,m,n)}{G_{odp}^{\tau}(r^l;b,\theta,m,n)}\}$$
$$\forall(\tau,p) \in alt_{od}(b,\theta,m,n),(\tau,p) \neq (\tau^*,p^*), \quad (12)$$
$$r_{odp^*}^{\tau^*,l+1}(b,\theta,m,n) = r_{odp^*}^{\tau^*,l}(b,\theta,m,n) + \psi_{od}^l(b,\theta,m,n), \quad (13)$$

where $\Delta_{odp}^{\tau,l}(b,\theta,m,n) = G_{odp}^{\tau}(r^l;b,\theta,m,n) - \pi_{od}(r^l;b,\theta,m,n)$ and

$$\psi_{od}^l(b,\theta,m,n) = \sum_{(\tau,p)\in alt_{od}(b,\theta,m,n)} \rho^l \times \frac{r_{odp}^{\tau,l}(b,\theta,m,n) \times \Delta_{odp}^{\tau,l}(b,\theta,m,n)}{G_{odp}^{\tau}(r^l;b,\theta,m,n)}.$$

This updating scheme is a multi-class adaptation of the path-swapping method often used in the literature (e.g., *21*, *22*, *23*) to solve for path flow-based traffic assignment problems. It implies a natural alternative flow adjustment mechanism: flows on the non-cheapest alternatives are moved to the least trip cost alternative and the volume moved out from a non-cheapest alternative is proportional to the relative difference between its trip cost and the least trip cost, which is intuitively based on the fact that travelers farther from the equilibrium and on alternatives with larger flow rates are more strongly inclined to change departure time and/or path than those on alternatives with smaller flow rates and with trip cost closer to the minimal cost.

4.4 Multi-class dynamic network loading (MDNL) using the traffic simulator

Consistent with the MSRDUE definition, all trips in a network are equilibrated in terms of *experienced* trip costs, consisting of experienced path times and path costs, so it is necessary to determine the experienced trip costs $G(r)$ for a given multi-class alternative flow vector r. To this end, the simulation-based dynamic traffic (network loading) model – DYNASMART (*10)* is employed to evaluate a given assignment r so as to obtain $G(r)$ and time-dependent link travel times used in the alterative generation step. DYNASMART adopts a hybrid (mesoscopic) approach to capture the dynamics of vehicular traffic flow in the simulation, whereby vehicles are moved individually according to prevailing local speeds, consistent with macroscopic flow relations on links. It should be noted that the MSRDUE algorithm presented in this paper is independent of the specific dynamic traffic model selected.

4.5 Convergence checking using gap values

Several criteria for convergence checking have been considered in the literature on DTA algorithms. For instance, Peeta and Mahmassani (*24*) adopted in their simulation-based DTA model a criterion based on the comparison of path assignments (or path flows) over successive iterations. This study extends the gap-based criterion (or measure) proposed in Lu (*19*) for the DUE problem to the RMCSRDUE context and defines the multi-class version of the gap function as follows:

$$Gap(r^l) = \sum_{u(b,\theta,m,n)} \sum_{o} \sum_{d} \sum_{(\tau,p)\in alt_{od}(b,\theta,m,n)} r_{odp}^{\tau,l}(b,\theta,m,n) \times \Delta_{odp}^{\tau,l}(b,\theta,m,n) \tag{14}$$

Note that, $Gap(r^l)$ provides a measure of the violation of the RMCSRDUE conditions in terms of the difference between the total actual experienced path generalized cost and the total least generalized cost evaluated at any given multi-class path flow pattern r. The difference vanishes when the path flow vector r^* satisfies the RMCSRDUE conditions. In the proposed solution algorithm, for practical considerations, if $|Gap(r^l) - Gap(r^{l-1})| \leq \varepsilon$ (a predetermined convergence threshold), convergence is assumed and the program goes back to the outer loop (step 2).

5 NUMERICAL EXPERIMENTS

A set of numerical experiments is conducted to examine the MSRDUE algorithm. The objectives are (i) to examine the algorithmic convergence and the solution quality of the algorithm and, (ii) with the explicit consideration of user heterogeneity, to investigate how the random parameters (i.e. VOT, VOESD, and VOLSD) in the MSRDUE model would affect departure time and path flow patterns (or toll road usage) under different dynamic pricing scenarios; that is, to compare the differences in departure time and path flow patterns between random parameter model and constant parameter model. The algorithm is coded and compiled using the Compaq Visual FORTRAN 6.6 and evaluated on the Windows XP platform and a machine with an Intel Pentium IV 2.8 GHz CPU and 2GB RAM. In all the experiments conducted, the following settings of the random parameters are applied. Note that the unit of VOT, VOESD, and VOLSD in this study is United States dollars (USD) per minute. The continuous distributions of VOT, VOESD, and VOLSD are assumed to follow, with no loss of generality, the (truncated) normal distribution, specified as follows:

VOT distribution: N(0.4, 0.2), $[\alpha^{min}, \alpha^{max}] = [0.01, 3.0]$;
VOESD distribution: N(0.3, 0.15), $[\beta^{min}, \beta^{max}] = [0.01, 2.0]$;
VOLSD distribution: N(1.8, 0.6), $[\lambda^{min}, \lambda^{max}] = [0.25, 4.0]$.

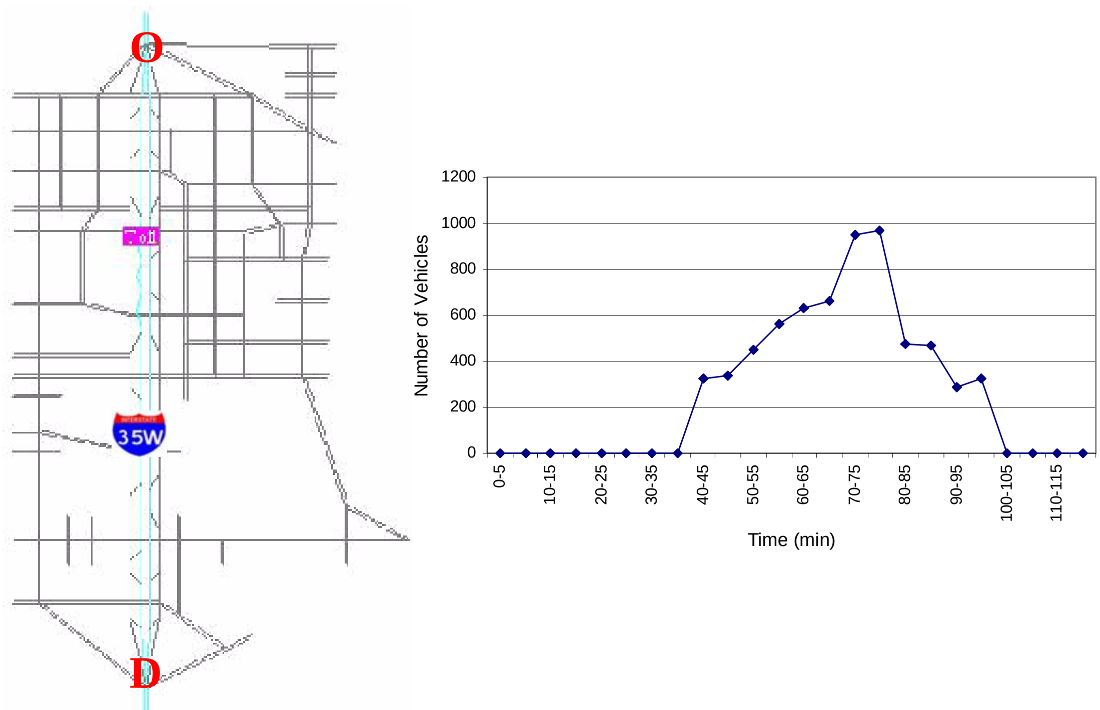
The VOT distribution is adapted from the estimated measurements in a value pricing experiment conducted in Southern California, USA (e.g., *24*, *25*), while the distributions of VOESD and VOLSD are determined by economic judgment based on the results reported in Small (*13*), due to the lack of estimated values from real world data. The resolution (aggregation interval) of the time-dependent shortest path tree calculation is set to 5 minutes, which is the same as the arrival time interval and the PAT interval. A strict convergence criterion is used in the inner loop of the column generation-based algorithm; that is $|Gap(r^l) - Gap(r^{l-1})|/Gap(r^l) \le 0.001$. The initial solutions of the experiments are obtained by loading given OD demands with a predetermined initial departure time distribution to the extreme efficient alternatives calculated based on prevailing travel costs output from the traffic simulator. Another measure of effectiveness (MOE) collected in the conducted experiments, in addition to the value of $Gap(r)$, is the average gap over all vehicles a the network for a given alternative flow pattern r.

$$AGap(r) = \frac{\sum_{u(b,\theta,m,n)} \sum_{o} \sum_{d} \sum_{(\tau,p)\in alt_{od}(b,\theta,m,n)} r_{odp}^{\tau,l}(b,\theta,m,n) \times \Delta_{odp}^{\tau,l}(b,\theta,m,n)}{\sum_{u(b,\theta,m,n)} \sum_{o} \sum_{d} \sum_{(\tau,p)\in alt_{od}(b,\theta,m,n)} r_{odp}^{\tau,l}(b,\theta,m,n)} \qquad (15)$$

This MOE is independent of problem size and thus useful for examining the convergence pattern and solution quality of the MSRDUE algorithm on different networks. The minimum of the $AGap(r)$ is zero. Essentially, the smaller the average gap, the closer the solution is to a BDUE.

Note that this study aims at developing a MSRDUE model for evaluating dynamic pricing scenarios, and not to solve for a toll vector that improves local or network-wide performance. Hence, the focus in the experiments conducted here is on demonstrating what the MSRDUE model can accomplish and why user heterogeneity should be addressed in evaluating dynamic road pricing scenarios as well as determining second-best pricing schemes, rather than to assess the effectiveness of specific dynamic toll vectors in reducing congestion.

The experiments are conducted on a portion of the Fort Worth (Texas, USA) network (Fig. 3(a)), consisting of 180 nodes (62 signalized nodes), 445 links and 13 traffic analysis zones (TAZ). A total of 25,500 vehicles are loaded onto the network over a 150-minute peak period. A critical OD pair (zone 1 to zone 2) is selected to examine the departure time and path flow patterns. This critical OD pair accounts for 25% of the total demand. The PAT distribution of those vehicles is shown in Fig. 3(b). To create the hypothetical pricing scenario, a toll road is added to the southbound freeway (I35W) corridor. The toll road is 3 miles long, while the general purpose road (i.e. original non-tolled freeway) is 4.5 miles long. Both roads have three lanes and the same performance function (e.g. capacity and speed limit). Table 1 lists the three dynamic pricing scenarios (i.e. low, mid, and high) tested in this experiment. The simulation planning horizon is 150 minutes.



(a) (b)

Fig. 3 Fort Worth network with hypothetical toll links and PAT pattern

Table 1 Dynamic road pricing scenarios tested on the Fort Worth network

Pricing Scenario	0-20 minutes	20-40 minutes	40-60 minutes	60-80 minutes	80-100 minutes	100-120 minutes	120-150 minutes
#1 (low)	$0.05	$0.20	$0.35	$0.50	$0.35	$0.20	$0.05
#2 (mid)	$0.25	$0.40	$0.55	$0.70	$0.55	$0.40	$0.25
#3 (high)	$0.45	$0.60	$0.75	$0.90	$0.75	$0.60	$0.45

We first examine the algorithmic convergence behavior of the MSRDUE algorithm in terms of average gap, number of schedule delay (SD) vehicles (early, late, and on-time vehicles), and departure time distributions, under dynamic pricing scenario #2. The convergence pattern in terms of average gap, defined in Eq.(15), is depicted in Fig. 4. This convergence pattern is compared with that of solving the constant parameter model (by the same algorithm), in which the parameters are set to equal to the means of the corresponding normal distributions assumed in the random parameter model (i.e. VOT = 0.4, VOESD = 0.3, and VOLSD = 1.8). It is shown in the figure that the convergence patterns of the solution algorithm for both models look similar, though the average gap values decrease non-monotonically. Moreover, the solution algorithm is able to find close-to-optimal solutions for both random parameter and constant parameter models as the final average gap values are very small (0.2-0.3 minutes) in both cases. Fig. 5 shows convergence pattern in terms of the number of SD vehicles of the critical OD pair in the random parameter model. As seen in the figure, the numbers of early and late vehicles decrease dramatically in the first few iterations, while the number of on-time vehicles increases steadily iteration by iteration (in reality, trip-makers generally tend to avoid penalties due to early or late arrival). The convergence pattern in terms of departure time distribution of the random parameter model is reported in Fig. 6. Although the departure times in the initial solution are evenly distributed between minutes 0 and 90 (an arbitrary starting point), the departure time pattern corresponding to the close-to-MSRDUE solution has an obvious peak between minutes 40 and 65, as trip-makers tend to depart before the toll charge is increased. This indicates that the mechanisms of alternative generation and flow equilibration of the algorithm are able to adjust the departure time pattern from disequilibrium to (near-) equilibrium.

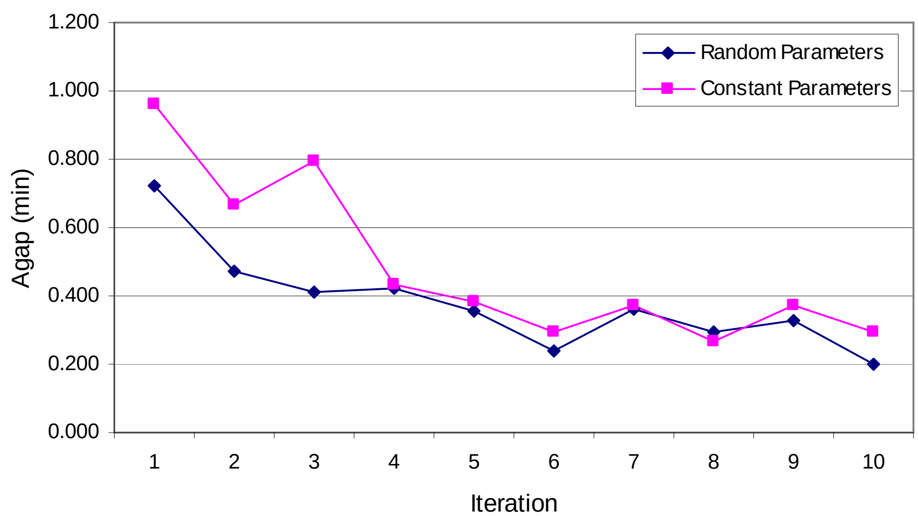


Fig. 4 Convergence pattern in terms of average gap on the Fort Worth network

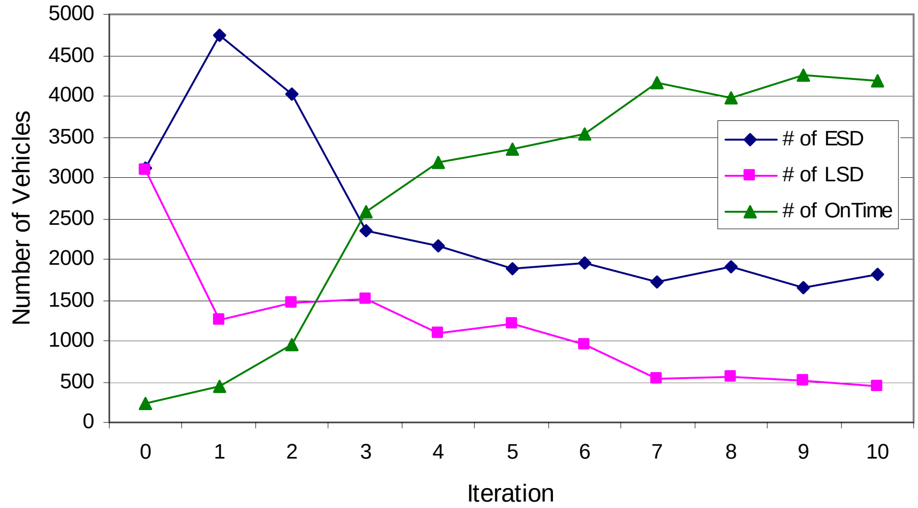


Fig. 5 Convergence pattern in terms of number of late, early and on-time vehicles for a critical OD pair on the Fort Worth network

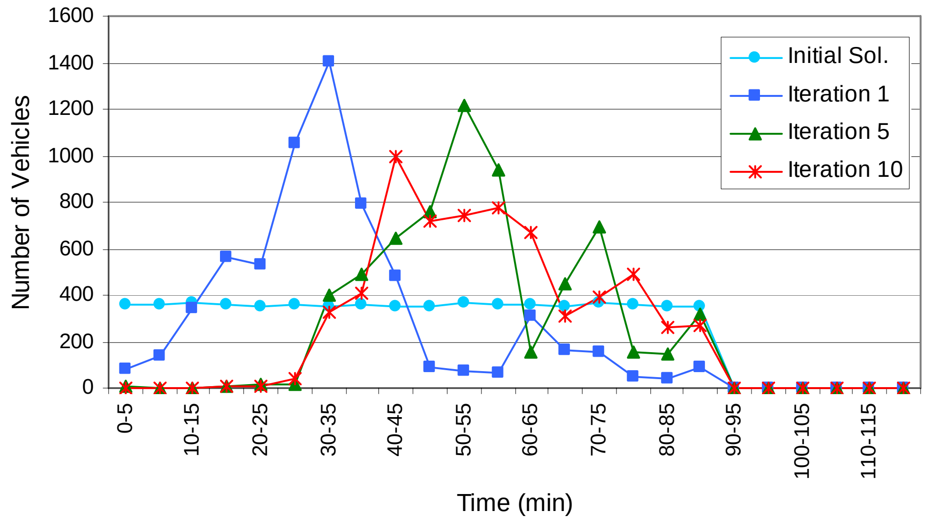


Fig. 6 Convergence pattern in terms of departure time distribution for a critical OD pair

A comparison of the critical OD pair's departure time patterns in the random parameter model and the constant parameter model is presented in Fig. 7. While the random parameter model predicts a departure time peak between 40 and 65 minutes, the constant parameter model anticipates a peak between 50 and 65 minutes. Furthermore, the central (peak) tendency of departure times in the constant parameter model is higher than that in the random parameter model. In summary, the peak of the departure time pattern is higher and occurs later under the constant parameter model than under the random parameter model. Similar observations can be found in the comparison of time-varying toll road usage (defined as the number of vehicles departing in each time interval and using the toll road) in the two models (Fig. 8). The constant parameter model also predicts higher toll road usage for this critical OD pair than the random parameter model (2991 versus 2436). These phenomena result from the constant VOT, VOESD, and VOLSD assumed in the model, under which trip-makers behave identically in choosing departure times and paths, whereas the random parameter model explicitly considers heterogeneous users with different VOT, VOESD, and VOLSD.

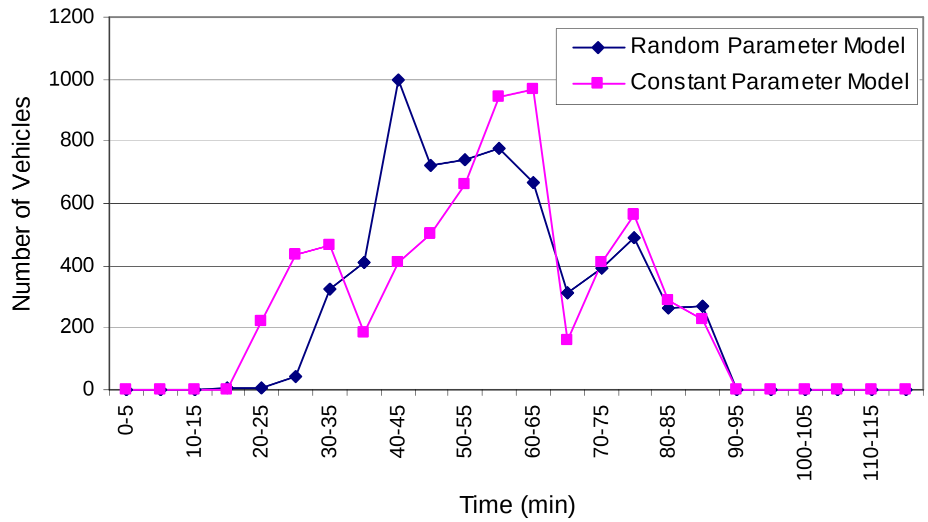


Fig. 7 Comparison of the departure time patterns obtained with the constant and random parameter models on the Fort Worth network

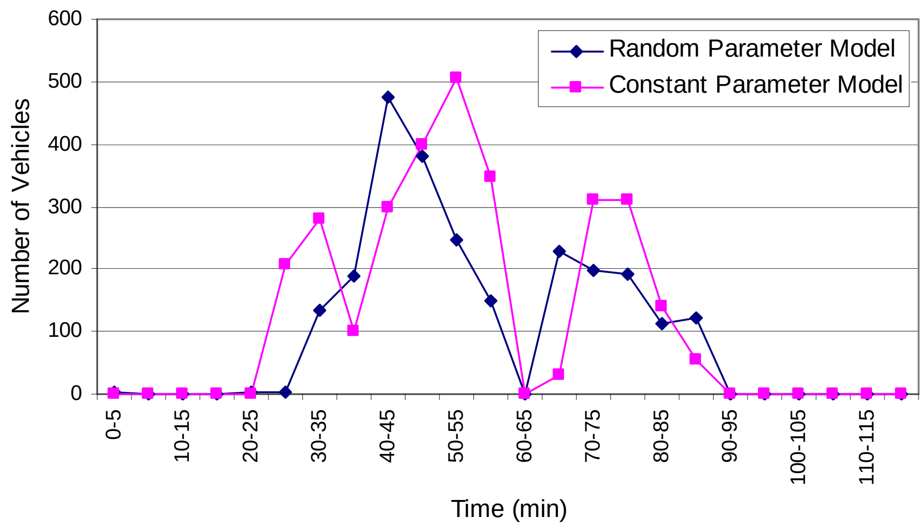


Fig. 8 Comparison of the time-varying toll road usage obtained with the constant and random parameter models on the Fort Worth network

The departure time patterns for that critical OD pair obtained with the random parameter model under the three different dynamic pricing scenarios (or levels) are presented in Fig. 9. As expected, when the toll charge is high, the departure time pattern shifts leftward, since the majority of trip-makers tends to depart earlier to avoid high tolls. On the other hand, the departure time pattern shifts rightward in the low price case, as most trip-makers are willing to use the cheap toll road to save travel time. These observations are also found in the comparison, depicted in Fig. 10, of the time-varying toll road usage of a critical OD pair obtained with the random parameter model under the different dynamic pricing scenarios. The numbers of vehicles using the toll road are 4323, 2436, and 1921 for the low price, mid price, and high price cases, respectively. Additionally, the peak of toll road usage is shifted to the time period between 20 – 45 minutes in the high price case, far earlier than the mid price and low price cases. These comparisons demonstrate that the proposed MSRDUE model and solution algorithm can effectively describe trip-makers' responses to time-varying toll charges in terms of temporal distribution (departure times) and spatial splits (path flows).

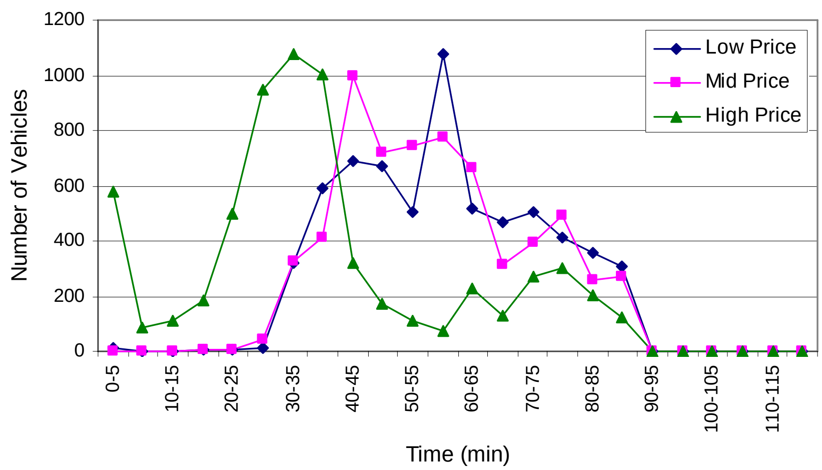


Fig. 9 Comparison of the departure time patterns obtained with the random parameter model under different dynamic pricing scenarios

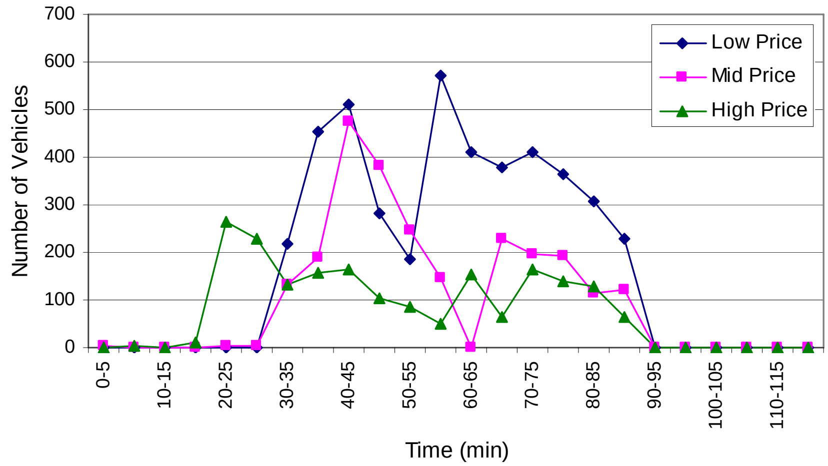


Fig. 10 Comparison of time-varying toll road usage obtained with the random parameter model under different dynamic pricing scenarios

6 CONCLUDING REMARKS

This paper presents a model and associated solution algorithm for the MSRDUE problem, which explicitly considers heterogeneous trip-makers with different behavioral characteristics, with time preferences reflected through the PAT (preferred arrival time), VOT (value of time), VOESD (value of early schedule delay), and VOLSD (value of late schedule delay), in determining users' responses to road pricing schemes (as well as other transport system measures) in a network. The model considers users' joint choice of departure time and path to the desired destination, so as to minimize their individual trip cost, which consists of a weighted sum of travel time, out-of-pocket cost, and schedule delay cost. As such the model can determine not only changes in route choice in response to dynamic pricing, but also temporal shifts towards less congested periods. The MSDUE problem is formulated as an infinite dimensional VI problem, and solved by a column generation-based algorithmic framework which embeds (i) an extreme non-dominated alternative-finding algorithm,SPAM, to obtain the VOT, VOESD, and VOLSD breakpoints that define multiple user classes, and determine the corresponding least cost alternative for each user class, (ii) a traffic simulator - DYNASMART in this application, to capture traffic dynamics and determine experienced path travel times; and (iii) a multi-class path flow equilibrating scheme to solve the reduced RMCSRDUE problem defined by a subset of feasible alternatives.

The experimental results show that the convergence patterns of the solution algorithm look similar for both the random parameter model and the constant parameter model, and the solution algorithm is able to find close-to-MSRDUE solutions as the final average gap values are very small. The results also show that the mechanisms for alternative generation and flow equilibration in the solution algorithm are able to adjust the departure time pattern from disequilibrium to (near-) equilibrium. As expected, there are significant differences in the respective departure time pattern and toll road usage predicted by the two models. Lastly, comparison of the departure time pattern and toll road usage for a critical OD pair obtained by the random parameter model under three different dynamic pricing scenarios (or levels) confirm that the proposed MSRDUE model and solution algorithm can realistically describe trip-makers' responses to time-varying toll charges in temporal distribution (departure times) and spatial splits (path flows).

Several interesting research directions can be continued based on the rich modeling capabilities of the MSRDUE model in capturing traffic dynamics and user heterogeneity. For instance, the model can be extended to consider OD-specific and/or time-varying PAT, VOT, VOESD, and VOLSD distributions, provided that the data are available to estimate the underlying parameter distributions. The model can also be integrated into a solution framework aiming at finding optimal dynamic pricing schemes, including locations, pricing periods and toll charges, so as to alleviate congestion. In addition, incorporating stochastic path choice with explicit perception errors (e.g. logit or probit models) would be an important and interesting extension. The model presented in this paper can be viewed as laying the foundation for a platform that integrates more realistic behavioral modeling in a dynamic network analysis tool. The main challenges in this development is to continue pushing the boundary of what can be realistically handled in a large network setting within the limits of practical computational capabilities.

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